

THE GEOMETRICAL THEORY OF HALOS—II<sup>1</sup>

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## PART 1. THE FUNDAMENTAL OPTICAL EQUATIONS

For completeness, the discussion will include a concise summary of the fundamental elementary optical laws upon which the calculations in the theory of halos must be based; their derivations will be found in standard treatises on optics or in other works to which reference will be made.

The only physical principles that are required in the purely geometrical problem of calculating the optical meteors that may be produced by crystals of a given form in a given orientation are the laws of simple refraction and simple reflection:

(1) In *simple refraction*, the incident and the refracted rays lie in the same plane with the normal at the point of incidence, while the sines of the angles of incidence and of refraction bear a constant ratio to one another:

$$\sin i = \mu \sin r.$$

The constant  $\mu$  is the index of refraction; if  $\mu > 1$ , the ray is turned toward the normal, otherwise away from the normal; only the former case need be considered, since the same computations may be applied when  $\mu < 1$  by interchanging the incident and the refracted rays (if  $\sin r$  becomes greater than unity, total internal reflection is indicated).

(2) When light is *regularly reflected* at an interface, externally or internally, the reflected ray lies in the plane through the incident ray and the normal to the interface at the point of incidence; the angle of incidence is equal to the angle of reflection:

$$i = R.$$

In both cases, the *deviation* of the ray, or angle through which it is turned from its original direction, is the angular displacement of the virtual image from the true position of an infinitely distant source.

## REFRACTION

An application of the law of refraction (fig. 1) at both the point of incidence and the point of emergence of a ray which traverses a prism gives the laws of prismatic refraction:

*Prismatic refraction in a principal plane.*—Consider first the case when the incident ray lies in the principal plane of a refracting dihedral angle, i. e., in the plane perpendicular to the refracting edge of the prism; the entire course of the ray is then in this plane. The general character of the path that will be followed at any given angle of incidence, figure 2, depends upon the relation between the value of the refracting angle  $\alpha$  and the maximum possible value of the angle of refraction (critical angle)  $\gamma = \arcsin(1/\mu)$ ; in the discussion of halos, it is usual to

adopt  $\mu = 1.31$  (the refractive index of ice for the yellow-green), whence  $\gamma = 49^\circ 45' 40''$ , and no light will be transmitted through two crystal faces inclined  $99^\circ 31'$  or more to each other. E. g., adjacent faces of a hexagonal prism do not constitute a refracting angle, but alternate faces form a truncated refracting angle of  $60^\circ$ . For the ultimate purpose of the present discussion, the best representation of the geometric relations involved is obtained by conceiving the prism to be placed at the center of a sphere of indefinitely great radius, to which all the lines and planes are extended, and then working with the resulting points and arcs on the sphere by means of spherical trigonometry, figures 2 and 3. With appropriate conventions of algebraic sign for the angles, as indicated in figure 2, the position of the image will in all cases be automatically given by the same set of formulae, figure 3;  $i$  is to be considered negative when the incident ray lies between the normal

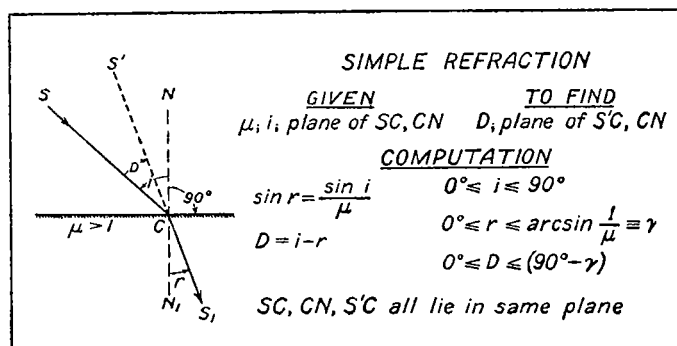


FIGURE 1. *Simple refraction.*—S, luminous source; SC, incident ray, lying in the optically rarer medium; CN, normal to interface at point of incidence;  $i$ , angle of incidence; CS<sub>1</sub>, refracted ray;  $r$ , angle of refraction; S', virtual image; D, deviation.

and the vertex, and a negative  $i'$  is to be interpreted as indicating a similar location for the emergent ray. As  $i$  varies between its extreme possible limits, the deviation D, or angular displacement of the image from the infinitely distant source, varies from a minimum

$$D_0 = 2 \arcsin \left( \mu \sin \frac{\alpha}{2} \right) - \alpha$$

at

$$\begin{cases} i = i' = \arcsin \left( \mu \sin \frac{\alpha}{2} \right) = \frac{1}{2} (D_0 + \alpha), \\ r = r' = \alpha/2, \end{cases}$$

to a maximum

$$D_m = 180^\circ - \{ \alpha + \arccos [\mu \sin (\alpha - \gamma)] \}$$

at both  $i = 90^\circ$  and  $i' = 90^\circ$ . The deviation of the image is always toward the position of the vertex of the refracting angle, V, which is  $90^\circ$  from N, in the plane of the face of incidence.

<sup>1</sup> For paper I, a general introductory discussion, see MON. WEATHER REV., 64:321-325, 1936.

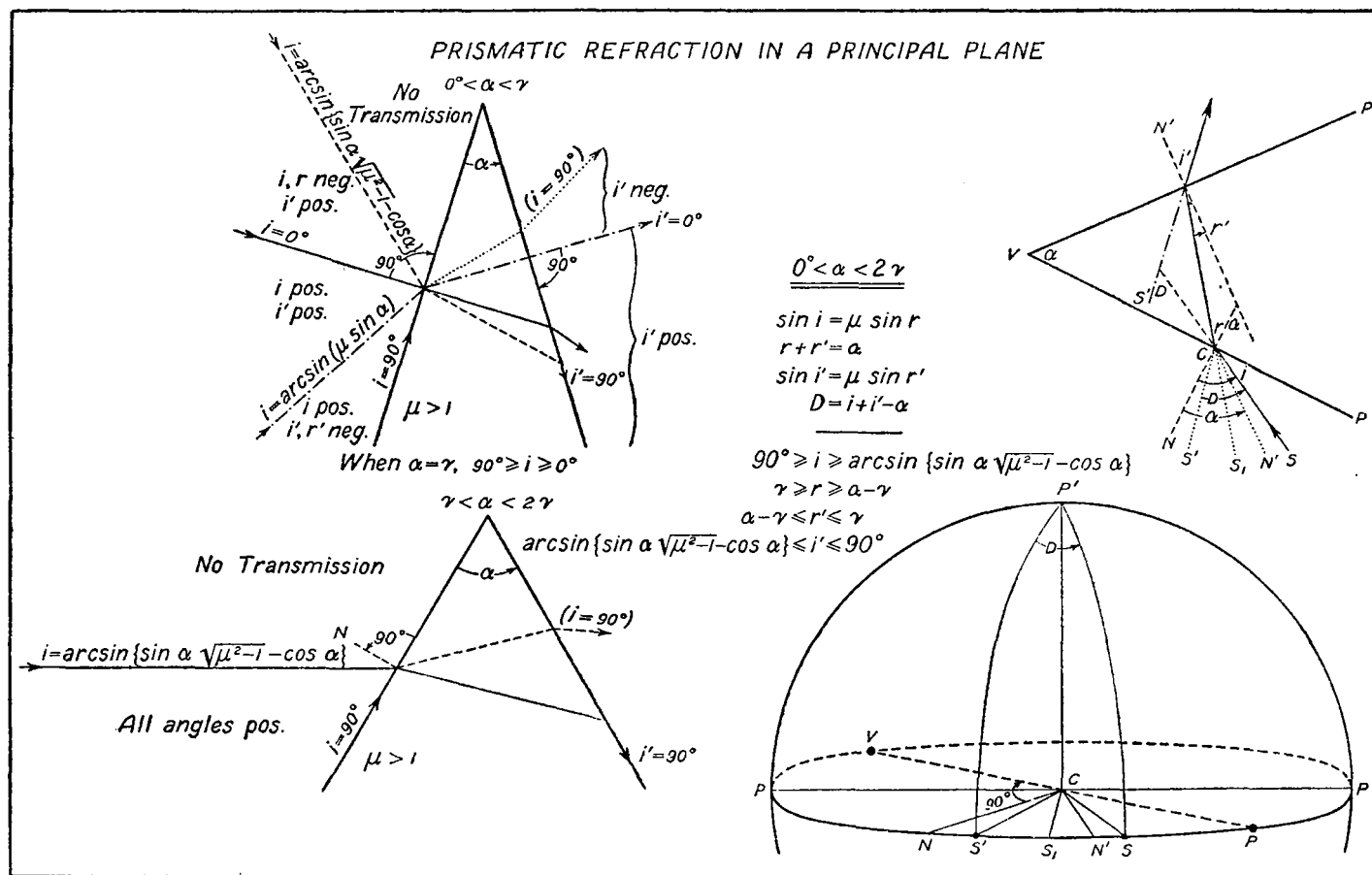
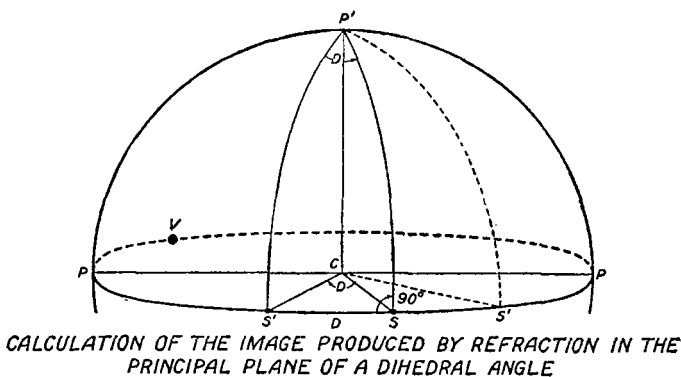


FIGURE 2. *Prismatic refraction in a principal plane.*—PP, principal plane; V, vertex of refracting angle  $\alpha$ ; VCP, face of incidence; P', pole of principal plane; N, N', normals at points of incidence and of emergence, respectively; CS<sub>1</sub>, internal ray. The deviation, D, is always toward the vertex.

*Oblique prismatic refraction.*—Now consider the case when the incident ray is inclined at an angle  $h$  to the principal plane, figure 4. It may be shown to follow from the law of refraction<sup>2</sup> that the emergent ray is inclined at the same angle,  $h$ , to the principal plane; and that the projection of the course of the ray onto the principal plane is exactly the path that would be followed by an actual ray in this plane if the index of refraction were  $\mu \frac{\cos k}{\cos h}$ , where  $k$  is given by  $\sin h = \mu \sin k$  and is the inclination of the internal ray to the principal plane. These relations are known as Bravais' Laws; and, with the preceding formulae for refraction in a principal plane, they provide means for the trigonometric calculation of the image produced by oblique prismatic refraction (fig. 5 and formulae I).

Because of the limitation which internal reflection puts on the angle of incidence in the principal plane, there is a limit to  $h$  beyond which no transmission takes place; in the case of a  $60^\circ$  refracting angle, e. g.,  $h$  cannot exceed  $60^\circ 45'$ . The deviation  $D$  is always less than  $D'$ ; the minimum of  $D$  is at that of  $D'$ , and the least minimum or *minimum minimorum* occurs when the ray traverses a symmetrical path in the principal plane. Only  $\alpha$ ,  $D$ ,  $D'$  may exceed  $90^\circ$ . The deviation  $D'$  is always toward the position of the vertex,  $V$ ,  $90^\circ$  from  $N$ , in the plane of the face of incidence. Several tables which facilitate computations with the preceding formulas accompany this paper.

<sup>1</sup> W. J. Humphreys, *Physics of the Air*, 2ed., New York, 1920, pp. 486-490. H. S. Uhler, *Amer. Math. Monthly*, 28: 1-10, 1921, *Amer. Jour. Sci.*, (4), 35: 389-423, 1913, and *Jour. Opt. Soc. Amer.*, 28: 89-90, 1936. Cf. L. Silberstein, *Vectorial Treatment of Refraction of Skew Rays by a Prism*, *Jour. Opt. Soc. Amer. and R. S. I.*, 16: 88-91, 1928; and M. Szulc, *Acta Physica Polonica*, 3: 115-121, 1934. The papers by Uhler form an especially complete and valuable discussion of prismatic refraction.



<u>GIVEN</u>	<u>TO FIND</u>
$\mu, \alpha, i$	Coordinates of $S'$ :
Coordinates of $S$ in	In Principal Plane System—
Principal Plane System:	Altitude $0^\circ$
Altitude $0^\circ$	Relative Azimuth $\pm D$
Relative Azimuth $0^\circ$	$S'$ Relative to $S$ —
	Deviation $D$
	Position Angle $\pm 90^\circ$

$$\begin{aligned} \sin r &= \frac{\sin i}{\mu} \\ r' &= \alpha - r \\ \sin i' &= \mu \sin r' \\ D &= i + i' - \alpha \end{aligned}$$

FIGURE 3. Calculation of the image produced by refraction in the principal plane of a dihedral angle.—PP, principal plane; P', pole of principal plane; V, vertex of refracting angle; S, source; S', image; D, deviation; C, observer.

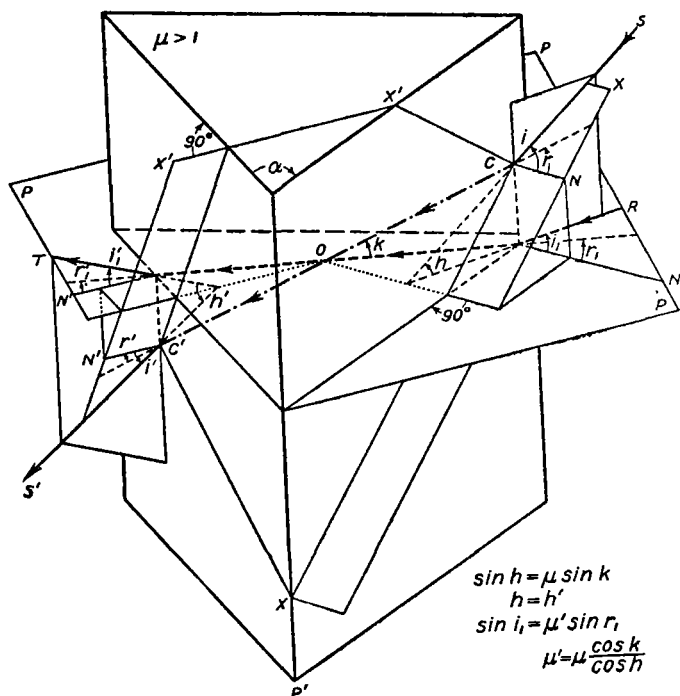


FIGURE 4. Oblique prismatic refraction.—SCC'S', path of ray; XX, plane of refraction at incidence; X'X', plane of refraction at emergence; PP, principal plane; CN, C'N', normals.

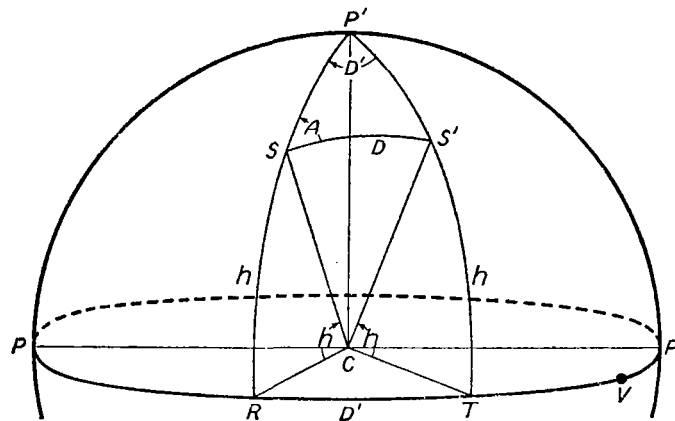


FIGURE 5. Calculation of the image produced by oblique prismatic refraction.—See Formulae I. PP, principal plane; P', pole of principal plane; V, vertex of refracting angle; S, source; S', image; D, deviation; A, position angle of image; h, inclination of incident ray to principal plane; R, T, projections of source and of image on principal plane; D', deviation of projection of ray on principal plane; C, observer. The deviation is always toward V.

### Formulae I

#### CALCULATION OF THE IMAGE PRODUCED BY SKEW PRISMATIC REFRACTION

Given	To Find
$\mu, \alpha; h, i_1$ .	Coordinates of S'
Coordinates of S:	Principal plane system:
Principal plane system:	Altitude $h$
Altitude $h$	Relative azimuth $\pm D'$
Relative azimuth $0^\circ$	Relative to S
(See table 1)	Deviation $D$
	Position angle $\pm A$

#### Computation

$$(1) \quad \mu' = \sqrt{\frac{\mu^2 - \sin^2 h}{1 - \sin^2 h}}, \quad 0^\circ \leq h \leq \arccos \left\{ \sqrt{\mu^2 - 1} \tan \frac{\alpha}{2} \right\} \quad (\text{Table 2})$$

$$(2) \quad \sin r_1 = \frac{\sin i_1}{\mu'}, \quad \arcsin \{ \sin \alpha \sqrt{\mu'^2 - 1} - \cos \alpha \} \leq i_1 \leq 90^\circ \quad (\text{Table 3})$$

$$(3) \quad r'_1 = \alpha - r_1 \quad (\text{Table 3})$$

$$(4) \quad \sin i'_1 = \mu' \sin r'_1 \quad (\text{Table 3})$$

$$(5) \quad D' = i_1 + i'_1 - \alpha \text{ toward position of vertex}$$

$$(6) \quad D = 2 \arcsin \left\{ \sin \frac{1}{2} D' \cos h \right\}, \quad D < D'$$

$$(7) \quad A = \arccot \left\{ \tan \frac{1}{2} D' \sin h \right\}$$

$$(8) \quad D'_0 = 2 \arcsin \left\{ \mu' \sin \frac{\alpha}{2} \right\} - \alpha \text{ at } i_1 = \frac{1}{2} (D'_0 + \alpha)$$

$$(9) \quad D'_m = 180^\circ - \left\{ \alpha + \arccos \left[ \mu' \sin \left( \alpha - \arcsin \frac{1}{\mu'} \right) \right] \right\} \text{ at } i_1 = 90^\circ$$

[See fig. 5. The first 5 formulae follow from Bravais' laws; and the next two from the solution of the right spherical triangle formed by dropping a perpendicular from the vertex P' of the isosceles triangle P'SS'.]